Appetizers:

1. Compute the least nonnegative integer congruent to $\sum_{i=1}^{2008}(i!)$ modulo 12.

2. A cylinder with a diameter of 12 cm is partially filled with water. Then a metal ball is dropped into this cylinder, sinking to the bottom and becoming completely submerged. If the water level rises 20 mm, find the radius of the ball. Be sure to include correct units.

3. John gave Paul 1/3 of his money. Then Paul gave Andrew 1/4 of the money he had after receiving money from John. Then Andrew gave John 1/10 of the money he had after receiving money from Paul. In the end, each had $90. How much money did each person have to start with?

4. Suppose $a_0, a_1, a_2, \ldots$ is an arithmetic sequence with $a_0 = 2$, $a_3 = a_1^2 - 8$, and $a_{15} > 0$. What must a_2 be?

5. Find a closed form expression for $f^{(k)}(x)$, the k^{th} derivative of $f(x)$, if

$$f(x) = \frac{1}{x^2 - 1}.$$

Entrees:

6. How many positive integers smaller than 10,000 are neither divisible by 6 nor by 9?

7. Two points, A and B, travel around a circle in the same direction with constant speeds. In one second, point A makes exactly one complete cycle, while point B makes 12. If A and B start in the same position, how many times during the first 10 seconds will they be positioned at antipodal points on the circle?

8. Find the area enclosed by the curve $y^2 = x^2 - x^4$.

9. A box contains four balls: one red, one green, and two blacks. Consider a game in which the player begins by drawing one ball from the box at random. If the red ball is drawn, the game is over and the player wins \$100. If either black ball is drawn, the game is over and the player wins nothing. If the green ball is drawn, the ball is returned to the bag and play continues. What is the probability that the player will win \$100 in this game?

10. For which real values of x does the power series $\displaystyle\sum_{n=2}^{\infty} \frac{12x^{2n}}{n(\ln n)^2}$ converge?

Desserts:

11. Suppose we are given a regular 14-gon, i.e. a regular polygon with 14 sides. How many convex quadrilaterals can be drawn, so that each side of the quadrilateral is a diagonal (and *not* a side) of the 14-gon?

12. Let $f(z)$ be the following real-valued function of a complex variable z.

$$f(z) = \frac{30}{|z^4 + 2iz|}$$

What is the maximum value of $f(z)$ over the circle where $|z| = 2$, and at what value or values does the maximum occur? You may give your answer in rectangular or polar form.

13. For how many positive integers, $x \leq 10,000$, is $2^x - x^2$ divisible by 7?

14. Suppose that we have an unlimited supply of poker chips in each of three colors: blue, red, and black. Find a closed expression for the number of ways that we can stack n poker chips in one column so that no two blue chips are touching.

15. Compute the derivative with respect to x of the following function when $x > 1$. Simplify as much as possible.

$$F(x) = \int_{\sqrt{\ln x}}^{\ln 2008} 2xe^{-t^2}\, dt$$

Appetizers:

1. Without a direct calculation involving limits, evaluate the following limit.

$$\lim_{t \to 0} \frac{\sin(\frac{\pi}{2} + t) - 1}{t}$$

2. How many **unordered** pairs of positive integers A, B are there whose least common multiple is $126,000$? (so (A, B) is considered the same as (B, A))

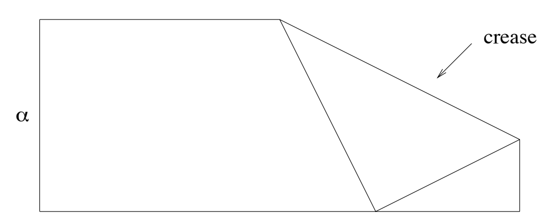
Entrees:

3. Let $B = B_n$ be the $n \times n$ matrix with -2's along the main diagonal, 1's directly above or below the main diagonal, and 0's everywhere else. In other words, $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} -2, & \text{if } i = j \\ 1, & \text{if } |i - j| = 1\text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Compute the determinant of B as a function of n. You must prove your answer.

4. A rectangular sheet of paper has width α and is very long. One corner of the paper is folded over so that it just touches the opposite side (see figure below). What is the minimum length of the crease which is formed?



Desserts:

5. Let $\theta = \cos^{-1}(1/\sqrt{3})$. Prove that $\frac{(\sqrt{3})^n}{\sqrt{2}} \sin(n\theta)$ is always an integer for all positive integers n.

6. Find all solutions to the differential equation given below. For full credit you must give your answer explicitly, not implicitly.

$$\frac{dy}{dt} = \sin(t + y)$$