## Appetizers:

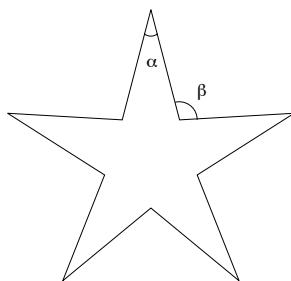**1.** Evaluate:

$$\int_1^4 \frac{\sqrt[3]{x} \cdot x^6}{x^7 \cdot \sqrt{x}} \, dx$$

**2.** The following figure is a *regular* pentagram, in the sense that all ten sides are congruent, all five interior angles are congruent to $\angle\alpha$, and all five exterior angles are congruent to $\angle\beta$. If the measure of $\angle\beta$ is $100°$, what is the measure of $\angle\alpha$?



**3.** If $n$ is a positive integer, let $r(n)$ denote the number obtained by reversing the order of the decimal digits of $n$. For example, $r(382) = 283$ and $r(410) = 14$. For how many two digit positive integers $n$ is the sum of $n$ and $r(n)$ a perfect square?

**4.** Find a positive real number such that the sum of that number and three times its reciprocal is as small as possible.

**5.** Find all $2 \times 2$ matrices with integer coefficients, whose determinant is at most 5, and which commute with the following matrix (under multiplication).

$$\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$$

## Entrees:

**6.** List all possible ways in which 72 can be written as the difference between two perfect squares. For example, $72 = 81 - 9$.

**7.** Let $g(x) = e^{x^2}$. Find $g^{(10)}(0)$, i.e., the $10^{\text{th}}$ derivative of $g$ evaluated at $x = 0$.

**8.** Persons $A$ and $B$ begin to play Ping Pong, a game at which they are **evenly matched**. Is it more likely that $A$ will win (exactly) 3 out of the first 4 games, or that $A$ will win (exactly) 5 out of the first 8 games? You must include the values of both probabilities in your answer.

**9.** Determine the number of three word phrases that can be formed from the letters in MATH ALL DAY. No "words" can be empty, and words do not have to make sense. For example, MAD HAT ALLY and T DMALL YAAH are valid phrases, but not HALT MALADY. You do not have to simplify your answer.

**10.** A symmetric monument is constructed so that it has a square base of width 10 ft and a height of 35 ft. Each edge of the base faces one of the four directions: North, South, East, or West. Moreover, the profile of the monument from any one of these directions is described by the function,

$$f(x) = \begin{cases} (x+6)^2 - 1, & \text{if } -5 \leq x \leq 0 \\ (x-6)^2 - 1, & \text{if } 0 < x \leq 5. \end{cases}$$

(see picture below). Set up, but **do not evaluate**, an integral which computes the total volume of this monument.



10 ft
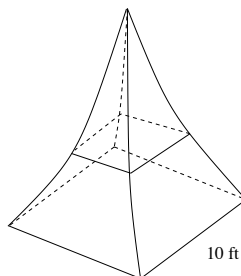
## Desserts:

**11.** Each of the three vertices of a triangle is connected by $n$ line segments to $n$ distinct points on the opposite side (none of which is another vertex). Assuming that no three segments intersect in the same point, into how many regions do these $3n$ line segments divide the interior of the triangle? For example, when $n = 1$ there are 7 regions, as pictured below.
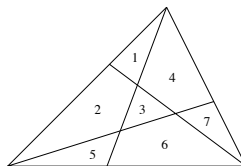


**12.** Define a function $f(x)$, whose domain is a subset of the real numbers, by setting

$$f(x) = \lim_{n \to \infty} \frac{2x}{1 + x^{2n}}$$

whenever the limit exists. Give a piecewise, i.e., split-domain, description for $f(x)$ which does not involve any limits. Determine all points at which $f(x)$ is continuous.

**13.** Find a closed form expression for the following sum.

$$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \cdots + \frac{2n+1}{n^2(n+1)^2}$$

**14.** Compute the following limit.

$$\lim_{x \to \infty} \frac{1}{xe^x} \int_{x^2}^{(x+1)^2} e^{\sqrt{t}} \, dt$$

**15.** How many positive integral solutions to the equation, $x + y + z = 30$, satisfy the property that the sum of the values of any two variables is at least as great as that of the third? Order matters here. So, for example, $(8, 11, 11)$ and $(11, 8, 11)$ are two distinct solutions.

**Hint:** Choosing a solution is equivalent to choosing $(x, y)$ such that $1 \leq x, y \leq 15$ and $15 \leq x + y < 30$.

## Appetizers:

**1.** Find an equation of the tangent line to the curve $y = \sin^3(5x)$ at the point where $x = \pi/15$, expressing the slope of the line as a rational number.

**2.** Fix an integer, $a \neq 1$. If $n$ is any positive integer, show that $a^{n+1} - n(a-1) - a$ is divisible by $(a-1)^2$.

## Entrees:

**3.** A housing developer converts an unused industrial property into 110 identical loft style apartments, which he initially offers for \$1500 per month. At this price, he is able to rent 100 units. However, when he raises the rent to \$1600 per month, he finds that he still is able to rent out 95 units (thereby increasing his gross income to $\$1600 \times 95 = \$152,000$).

    **a.** Assuming that the number of occupied units, $n(p)$, is a **linear** function of the price, how much should the developer charge in order to maximize his gross income?

    **b.** Assuming that $n$ varies **exponentially** with $p$, so that it decreases by 5% with every increase in price of \$100, how much should the developer charge in order to maximize his gross income? **Note:** Your answer will have a natural log in it.

**4.** Consider the infinite series

$$\frac{1}{1} + \frac{1}{10} + \frac{1}{11} + \frac{1}{100} + \frac{1}{101} + \frac{1}{110} + \frac{1}{111} + \ldots$$

whose terms are the reciprocals of all positive integers which have only 0's and 1's as digits (taken in the implied order). Does this series converge or diverge? You must explain your answer.

## Desserts:

**5.** Let $f(x) = x + 1$ and $g(x) = 1/x$. Say that a positive rational number $p/q$ is **representable**, if $h(1) = p/q$ for some $h(x)$ which can be written as the composition of finitely many $f$'s and $g$'s. For example, $8/3$ is representable since $f \circ f \circ g \circ f \circ g \circ f(1) = 8/3$.

    **a.** Show that $38/17$ is representable.

    **b.** Show that *every* positive rational number is representable.

**6.** Direct wired communication lines are laid down between each of four military outposts: $O_1$, $O_2$, $O_3$, and $O_4$. Each communication line either survives or fails independently of the others, and each line has the same probability, $p$, of surviving a day. When a direct line between two outposts fails, the communication between them continues as long as there is an uninterrupted path of multiple lines through other outposts. Find the probability that at the end of the day, communication between outpost $O_1$ and $O_4$ is still possible.