Appetizers:

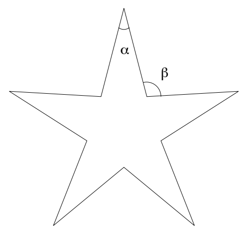
1. Evaluate:

$$\int_1^4 \frac{\sqrt[3]{x} \cdot x^6}{x^7 \cdot \sqrt{x}} \, dx$$

Solution:

$$\int_1^4 x^{-7/6} \, dx = -6x^{-1/6} \Big|_1^4 = \frac{-6}{\sqrt[3]{2}} + 6$$

2. The following figure is a *regular* pentagram, in the sense that all ten sides are congruent, all five interior angles are congruent to $\angle\alpha$, and all five exterior angles are congruent to $\angle\beta$. If the measure of $\angle\beta$ is $100°$, what is the measure of $\angle\alpha$?



Solution: Inscribe the figure in a regular pentagon. Each interior angle of the regular pentagon must have measure $\frac{1}{5}(180)(5-2) = 108$ degrees. Thus, we have the equation:

$$m\angle\alpha + 2\left(\frac{180 - m\angle\beta}{2}\right) = 108.$$

So $\angle\alpha$ must have measure $100 + 108 - 180 = 28$ degrees.

3. If n is a positive integer, let $r(n)$ denote the number obtained by reversing the order of the decimal digits of n. For example, $r(382) = 283$ and $r(410) = 14$. For how many two digit positive integers n is the sum of n and $r(n)$ a perfect square?

Solution: Let a and b be the digits of n, so that $n = 10a + b$, $0 \le a, b \le 9$ and $a \ne 0$. Then

$$n + r(n) = 10a + b + 10b + a = 11(a + b).$$

So the key observation is that $n + r(n)$ can only be a perfect square if 11 divides $a + b$. This leads to the following complete list of solutions:

$$n \in \{29, 38, 47, 56, 65, 74, 83, 92\}.$$

4. Find a positive real number such that the sum of that number and three times its reciprocal is as small as possible.

Solution: We want to minimize the function, $f(x) = x + 3/x$, over the positive real numbers. The First Derivative Test with $f'(x) = 1 - 3/x^2$ shows us that f *does* have a minimum value over this domain, and it occurs at the unique positive critical point where $x = \sqrt{3}$ and $f(\sqrt{3}) = 2\sqrt{3}$.

5. Find all 2×2 matrices with integer coefficients, whose determinant is at most 5, and which commute with the following matrix (under multiplication).

$$\begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$$

Solution: Call the given matrix M and let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Setting $AM = MA$ immediately leads to an equivalent set of relations on the coefficients of A, namely $a = d$ and $b = -3c$. So the matrices which commute with M are precisely those of the form

$$\begin{bmatrix} a & -3c \\ c & a \end{bmatrix}.$$

Since the determinant, $a^2 + 3c^2$, must be at most 5, A can be any matrix of the above form in which

$$(a, c) \in \{(0,0), (0,\pm 1), (\pm 1, 0), (\pm 1, \pm 1), (\pm 2, 0)\}.$$

Entrees:

6. List all possible ways in which 72 can be written as the difference between two perfect squares. For example, $72 = 81 - 9$.

Solution: Suppose $x^2 - y^2 = 72$, for positive integers x and y with $x > y$. (Note: there is no need to consider negative values for x and y, as $(-a)^2 = a^2$). Then $(x+y)(x-y)$ is a factorization of 72. Moreover, (x, y) is completely determined by this factorization, since

$$\begin{cases} x + y = m \\ x - y = n \end{cases}$$

has the unique solution: $x = (m+n)/2$, $y = (m-n)/2$. Therefore, we can find all solutions by solving the above system for every possible factorization of 72 as mn where $m > n > 0$. Doing this, we get **integral** solutions for x and y precisely when $(m, n) = (36, 2)$, $(18, 4)$, and $(12, 6)$. In particular, we get $(x, y) = (19, 17)$, $(11, 7)$, and $(9, 3)$ respectively. Thus, the only three ways to write 72 as the difference of perfect squares are:

$$72 = 361 - 289 = 121 - 49 = 81 - 9.$$

7. Let $g(x) = e^{x^2}$. Find $g^{(10)}(0)$, i.e., the 10^{th} derivative of g evaluated at $x = 0$.

Solution: The Taylor Series for e^x at $x = 0$ converges (to e^x) for all x and is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

Therefore, we can obtain the Taylor series for e^{x^2} by substituting x^2 in for x.

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!} + \cdots$$

Now, since the coefficient of x^{10} must be $g^{10}(0)/10!$, this implies that $g^{10}(0) = 10!/5!$.

8. Persons A and B begin to play Ping Pong, a game at which they are **evenly matched**. Is it more likely that A will win (exactly) 3 out of the first 4 games, or that A will win (exactly) 5 out of the first 8 games? You must include the values of both probabilities in your answer.

Solution: For the first 4 games played, the probability space has $2^4 = 16$ equally likely outcomes, such as $ABAA$ and $BAAB$. There are 4 favorable outcomes, however, since there are 4 rearrangements of $AAAB$. Thus, the probability of A winning exactly 3 of the first 4 is $4/16$ or $1/4$.

For the first 8 games, there are 2^8 equally likely outcomes. If we want A to win exactly 5 of those games, there are $C(8, 5) = 8!/(5!\,3!)$ favorable outcomes. Therefore the probability is given by:

$$\frac{C(8,5)}{2^8} = \frac{8!}{5!\,3!\,2^8} = \frac{8 \cdot 7}{2^8} = \frac{7}{32}.$$

It's more likely that A will win exactly 3 of the first 4 games.

9. Determine the number of three word phrases that can be formed from the letters in MATH ALL DAY. No "words" can be empty, and words do not have to make sense. For example, MAD HAT ALLY and T DMALL YAAH are valid phrases, but not HALT MALADY. You do not have to simplify your answer.

Solution: View the overall task of forming a three word phrase as a two step process. First, we arrange the 10 letters. Then, we decide where the two spaces go. By the Product Rule for Counting, the total number of ways to form a three word phrase will be the product of the numbers of ways to do these two smaller tasks.

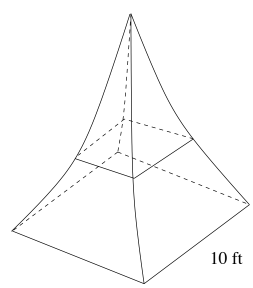
Because there are three A's and two L's, the number of ways to arrange the letters is $10!/(3!\,2!)$. Then we have $C(9,2) = 9!/(2!\,7!)$ ways to choose the two gaps (out of nine) where the spaces must go. So the total number of three word phrases is given by

$$\frac{10!}{3!\,2!} \cdot \frac{9!}{2!\,7!} = 3 \cdot 10! = 10{,}886{,}400.$$

10. A symmetric monument is constructed so that it has a square base of width 10 ft and a height of 35 ft. Each edge of the base faces one of the four directions: North, South, East, or West. Moreover, the profile of the monument from any one of these directions is described by the function,

$$f(x) = \begin{cases} (x+6)^2 - 1, & \text{if } -5 \le x \le 0 \\ (x-6)^2 - 1, & \text{if } 0 < x \le 5. \end{cases}$$

(see picture below). Set up, but **do not evaluate**, an integral which computes the total volume of this monument.



Solution: We will compute the volume by integrating the cross-sectional area function of the height z. These cross-sections are squares, such that the length of any side is given by

$$s(z) = (6 - \sqrt{z+1}) - (-6 + \sqrt{z+1}) = 12 - 2\sqrt{z+1}.$$

Therefore the total volume is given by

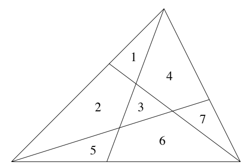
$$\int_0^{35} A(z)\,dz = \int_0^{35} (s(z))^2\,dz = \int_0^{35} 4(6 - \sqrt{z+1})^2\,dz.$$

Alternatively, we can place the base on the xy-plane in $\mathbb{R}^3$, such that the x and y axes point in the four directions. This leads to the following triple integral for the volume of the monument.

$$\int_0^{35} \int_{-6+\sqrt{z+1}}^{6-\sqrt{z+1}} \int_{-6+\sqrt{z+1}}^{6-\sqrt{z+1}} 1\,dx\,dy\,dz$$

Desserts:

11. Each of the three vertices of a triangle is connected by n line segments to n distinct points on the opposite side (none of which is another vertex). Assuming that no three segments intersect in the same point, into how many regions do these $3n$ line segments divide the interior of the triangle? For example, when $n = 1$ there are 7 regions, as pictured below.



Solution: When n lines are drawn from the first vertex, this divides the interior into $(n+1)$ regions. Then, each segment drawn from the second vertex creates a "new" region (divides some region into two, really) every time it crosses a line, including the opposite side. Since it must cross all n of the segments from the first vertex, the number of regions in the interior after drawing all $2n$ segments from the first two vertices is

$$(n+1) + n(n+1) = (n+1)^2.$$

Finally, each segment drawn from the third vertex crosses all $2n$ of the segments from the other vertices. By the same reasoning as before, this creates $2n+1$ "new" regions in the interior. So the total number of the regions in the interior is given by

$$(n+1)^2 + n(2n+1) = 3n^2 + 3n + 1.$$

12. Define a function $f(x)$, whose domain is a subset of the real numbers, by setting

$$f(x) = \lim_{n\to\infty} \frac{2x}{1+x^{2n}}$$

whenever the limit exists. Give a piecewise, i.e., split-domain, description for $f(x)$ which does not involve any limits. Determine all points at which $f(x)$ is continuous.

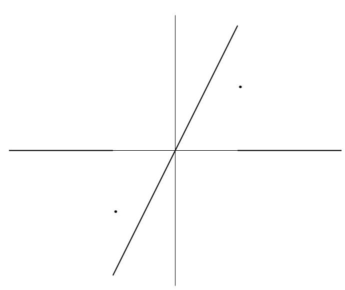
Solution: A good first step is to compute the following limit.

$$\lim_{n\to\infty} x^{2n} = \begin{cases} 0, & \text{if } |x| < 1 \\ 1, & \text{if } |x| = 1 \\ \infty, & \text{if } |x| > 1 \end{cases}$$

Remembering that x is a constant with respect to the limit, this gives us:

$$f(x) = \lim_{n\to\infty} \frac{2x}{1+x^{2n}} = \begin{cases} 2x, & \text{if } |x| < 1 \\ x, & \text{if } |x| = 1 \\ 0, & \text{if } |x| > 1. \end{cases}$$

The function is continuous everywhere except $x = \pm 1$, and looks something like the following.



13. Find a closed form expression for the following sum.

$$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \cdots + \frac{2n+1}{n^2(n+1)^2}$$

Solution: First break down the terms with partial fraction decomposition:

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{(n+1)} + \frac{D}{(n+1)^2}.$$

Solving for the coefficients, we find that $B = 1$, $D = -1$, and $A = C = 0$. So the sum telescopes.

$$\left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = 1 - \frac{1}{(n+1)^2}$$

14. Compute the following limit.

$$\lim_{x\to\infty} \tfrac{1}{xe^x} \int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt$$

Solution: Applying L'Hospital's Rule, FTC, and the Chain Rule, we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{\int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt}{xe^x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt}{\frac{d}{dx}(xe^x)} \\
&= \lim_{x\to\infty} \frac{2(x+1)e^{x+1} - 2xe^x}{xe^x + e^x} \\
&= \lim_{x\to\infty} \frac{2e(x+1) - 2x}{x+1} = 2e - 2.
\end{aligned}$$

Note: One should first verify that L'Hospital applies (which it does). Also, it *is* possible to do a u-substitution and simply work out the integral explicitly before taking the limit.

15. How many positive integral solutions to the equation, $x + y + z = 30$, satisfy the property that the sum of the values of any two variables is at least as great as that of the third? Order matters here. So, for example, $(8, 11, 11)$ and $(11, 8, 11)$ are two distinct solutions.

Hint: Choosing a solution is equivalent to choosing (x, y) such that $1 \le x, y \le 15$ and $15 \le x + y < 30$.

Solution: Let $s(x)$ be the number of solutions for each x. For example, $s(1) = 2$, since y can be 14 or 15 when $x = 1$, leading to the solutions $(1, 14, 15)$ and $(1, 15, 14)$ respectively. Similarly, $s(2) = 3$, since y can be 13, 14, or 15 when $x = 2$, leading to the solutions $(2, 13, 15)$, $(2, 14, 14)$, and $(2, 15, 13)$. More generally, for any $x \le 14$, $s(x) = x + 1$, as the bounds on y are given by $15 - x \le y \le 15$. The formula does *not* apply when $s(15) = 14$, because at that point the conditions, $x + y < 30$ and $y > 0$, become relevant. By summing $s(x)$ over all x values, the total number of solutions is

$$\begin{aligned}
\sum_{x=1}^{15} s(x) &= 14 + \sum_{x=1}^{14} s(x) \\
&= 14 + \sum_{x=1}^{14} x + \sum_{x=1}^{14} 1 \\
&= 14 + \frac{14 \cdot 15}{2} + 14 = 133.
\end{aligned}$$

Appetizers:

1. Find an equation of the tangent line to the curve $y = \sin^3(5x)$ at the point where $x = \pi/15$, expressing the slope of the line as a rational number.

Solution: Let $f(x) = \sin^3(5x)$. Then, by the Chain Rule,

$$f'(x) = 3\sin^2(5x) \cdot \cos(5x) \cdot 5 = 15\sin^2(5x)\cos(5x).$$

So the slope of the tangent line when $x = \pi/15$ is given by

$$f'(\pi/15) = 15\sin^2(\pi/3)\cos(\pi/3) = 15(\sqrt{3}/2)^2(1/2) = 45/8.$$

To find an equation for the line, we use this slope and the point, $(\pi/15, 3\sqrt{3}/8)$, to get the point-slope form:

$$y - 3\sqrt{3}/8 = (45/8)(x - \pi/15).$$

2. Fix an integer, $a \neq 1$. If n is any positive integer, show that the integer $a^{n+1} - n(a-1) - a$ is divisible by $(a-1)^2$.

Solution: Think of $p(a) := a^{n+1} - n(a-1) - a$ as a polynomial in $\mathbb{Z}[a]$. By Division Algorithm, Gauss' Lemma, and the fact that $p(1) = 0$, we know that $p(a) = (a-1)q(a)$ for some polynomial $q(a) \in \mathbb{Z}[a]$. Now differentiate both sides and plug in $a = 1$ again.

$$\begin{aligned} (a-1)q'(a) + q(a) = p'(a) &= (n+1)a^n - n - 1 \\ 0 + q(1) = p'(1) &= (n+1)(1) - n - 1 = 0 \end{aligned}$$

So now it follows that $q(a) = (a-1)\hat{q}(a)$ for some $\hat{q}(a) \in \mathbb{Z}[a]$. Thus, $(a-1)^2 \big| p(a)$ in $\mathbb{Z}[a]$, which implies the same result over $\mathbb{Z}$ if a is replaced with any integer.

A second solution is to prove the statement by a straightforward induction argument. When $n = 1$, the statement says that

$$(a-1)^2 \big| a^2 - 2a + 1,$$

which is clearly true. Then the inductive step follows from the calculation:

$$a^{n+2} - (n+1)(a-1) - a = a(a^{n+1} - n(a-1) - a) + (n+1)(a-1)^2.$$

Entrees:

3. A housing developer converts an unused industrial property into 110 identical loft style apartments, which he initially offers for \$1500 per month. At this price, he is able to rent 100 units. However, when he raises the rent to \$1600 per month, he finds that he still is able to rent out 95 units (thereby increasing his gross income to $\$1600 \times 95 = \$152,000$).

a. Assuming that the number of occupied units, $n(p)$, is a **linear** function of the price, how much should the developer charge in order to maximize his gross income?

Solution: Using point slope form and the two data points, $(1500, 100)$ and $(1600, 95)$, we obtain the following models for number of units rented and gross income.

$$\begin{aligned} n(p) &= -\frac{1}{20}(p-1500)+100 \\ I(p) = p \cdot n(p) &= -\frac{p}{20}(p-1500)+100p. \end{aligned}$$

In order to maximize $I(p)$, we differentiate once to get $I'(p) = 175 - p/10$. From this we see that there is a maximum value for $I(p)$, and it occurs when $I(\$1750) = \$153,125$.

b. Assuming that n varies **exponentially** with p, so that it decreases by 5% with every increase in price of \$100, how much should the developer charge in order to maximize his gross income? (Note: Your answer will have a natural log in it.)

Solution: Using an exponential model, we arrive at the following equations for $n(p)$ and $I(p)$.

$$\begin{aligned} n(p) &= 100(.95)^{\left(\frac{p-1500}{100}\right)} = 100e^{\ln(.95)(.01p-15)} \\ I(p) &= 100p(.95)^{\left(\frac{p-1500}{100}\right)} = 100pe^{\ln(.95)(.01p-15)} \end{aligned}$$

Once again, we compute $I'(p)$ and set it equal to 0 to find where the maximum occurs.

$$\begin{aligned} I'(p) &= 100p \cdot e^{\ln(.95)(.01p-15)}(.01\ln(.95)) + 100 \cdot e^{\ln(.95)(.01p-15)} \\ &= e^{\ln(.95)(.01p-15)}(p\ln(.95)+100) \end{aligned}$$

Since $\ln(.95) < 0$, this shows us that there *is* a maximum, and that it occurs when $p = -100/\ln(.95) \approx \$1950$. Under this model, the developer can expect a maximum gross income of about \$154,807.

4. Consider the infinite series

$$\frac{1}{1}+\frac{1}{10}+\frac{1}{11}+\frac{1}{100}+\frac{1}{101}+\frac{1}{110}+\frac{1}{111}+\dots$$

whose terms are the reciprocals of all positive integers which have only 0's and 1's as digits (taken in the implied order). Does this series converge or diverge? You must explain your answer.

Solution: First note that there are 2^{n-1} numbers which have n digits, each of which is a 0 or 1. So modulo grouping of the terms, this series is "less than" the series,

$$\frac{1}{1}+2\cdot\frac{1}{10}+4\cdot\frac{1}{100}+8\cdot\frac{1}{1000}+\cdots.$$

This new series converges (to $1/(1-1/5) = 5/4$), as it is a geometric series with ratio $1/5$. Therefore, by the Comparison Test, the original series must also converge. If one is troubled by the grouping of terms, there are a variety of ways to make the justification completely rigorous. For example, note that each partial sum of the original series is less than a partial sum of the geometric series, which is in turn less than $5/4$. Thus, the sequence of partial sums of the original is a bounded sequence, and clearly monotone increasing as all terms in the series are non-negative. Therefore, the Monotone Convergence Theorem would imply convergence of the original series.

Desserts:

5. Let $f(x) = x + 1$ and $g(x) = 1/x$. Say that a positive rational number p/q is **representable**, if $h(1) = p/q$ for some $h(x)$ which can be written as the composition of finitely many f's and g's.

For example, $8/3$ is representable since $f \circ f \circ g \circ f \circ g \circ f(1) = 8/3$.

a. Show that $38/17$ is representable.

Solution: Working backwards, $38/17 = 4/17 + 2$, and $17/4 = 1/4 + 4$. So $38/17$ is representable since

$$f \circ f \circ g \circ f \circ f \circ f \circ f \circ g \circ f \circ f \circ f(1) = 38/17.$$

b. Show that *every* positive rational is representable.

Solution: We will prove this by strong induction on the denominator, q, when p/q is in lowest terms. Let $S(n)$ be the statement that every positive rational number, $r = p/q$, which is in lowest terms, and for which $q \le n$, is representable. The statement is clearly true for $n = 1$, since all positive integers can be obtained from 1 by repeated application of f. So now suppose that $S(n)$ is true for all $1 \le n < n_0$, for some $n_0 > 1$. We want to show $S(n_0)$.

Let $r = p/q$ be in lowest terms and suppose $q \le n_0$. If $q < n_0$, then r is representable by inductive hypothesis. If $q = n_0$, write $r = m + p_0/q$ where m is a non-negative integer and $p_0 < q$ (i.e., write as a mixed number). Note that p_0 can not be 0 since p/q was in lowest terms. Therefore, q/p_0 has denominator less than n_0. So by inductive hypothesis, q/p_0 is representable as $h(1)$ for some h. Now just take $\hat{h}(x) = f^m \circ g \circ h(x)$. Then

$$\hat{h}(1) = f^m \circ g \circ h(1) = f^m \circ g(q/p_0) = m + p_0/q = r.$$

Thus, we have shown $S(n_0)$ using only the inductive hypothesis. Along with the base case and the Principle of Strong Induction, this shows $S(n)$ for all $n \ge 1$, i.e., every positive rational number is representable.

6. Direct wired communication lines are laid down between each of the four military outposts: O_1, O_2, O_3, and O_4. Each communication line either survives or fails independently of the others. Each line has the same probability, p, of surviving a day. When a direct line between two outposts fails, the communication between them continues as long as there is an uninterrupted path of multiple lines through other outposts. Find the probability that at the end of the day the communication between outpost O_1 and O_4 is still possible.

Solution: There are six communication lines to consider. So the probability of any *particular* outcome which has exactly k lines intact is $p^k(1-p)^{6-k}$. Hence, one option would be to enumerate all $2^6 = 64$ possible configurations, identify which ones leave O_1 and O_4 connected, and then sum up the probabilities of these outcomes using the above formula. Ideally, however, we would consider classes of symmetric cases, in order to cut down on the enumeration.

We will first compute the probabilities of those configurations which do **not** leave O_1 and O_4 connected, and classify by number of connected components in the graph. There will be four connected components precisely if all six lines fail, for a total probability of $(1-p)^6$. There will be three connected components whenever only one line survives. However, that one can **not** be the line between O_1 and O_4. So here we have a total probability of $5p(1-p)^5$. By similar reasoning, the total probability of two connected components with two elements each (still with O_1 and O_4 disconnected) is $2p^2(1-p)^4$. Finally, for two connected components with 3 elements in one and 1 element in the other is

$$2(p^3(1-p)^3 + 3p^2(1-p)^4).$$

Summing up the probabilities of these disjoint cases, and then subtracting from 1 to get the probability that O_1 and O_4 *are* still connected, we get

$$p + 2p^2 - 7p^4 + 7p^5 - 2p^6$$

as our final answer.