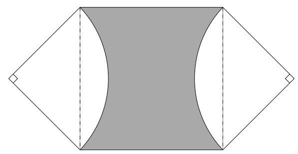
Appetizers:

1. The figure below shows a unit square, partially covered by two quarter-circles. What is the area of the shaded region? Be sure to simplify your answer.



2. Write the number 2010 in base 3.

3. A function $y = f(x)$ is implicitly defined by the equation, $3^y = x^2$. Determine the derivative, $\frac{dy}{dx}$, at the point $(9, 4)$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies $f(-5) = 8$ and $f(0) = 2$, and is **even**. Define a new function g by

$$g(x) = \begin{cases} f(x), & \text{if } x \leq 0 \\ 4 - f(x), & \text{if } x > 0 \end{cases}.$$

Compute $\displaystyle\int_{-5}^{5} g(x)\, dx$.

5. For distinct primes p and q, and a positive integer n, define the representation number $C_{p,q}(n)$ to be the number of integral solutions (x, y, z) to the equation

$$x^2 + py^2 + qz^2 = n.$$

Compute $C_{3,7}(56)$.

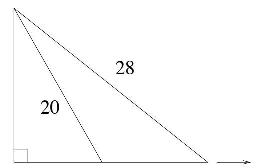
Entrees:

6. How many solutions does the following equation have over the closed interval $[-2\pi, 2\pi]$?

$$2 - \sin^2\theta = 3\cos\theta - \cos^3\theta$$

7. Let D_a be the region in the xy-plane which is bounded by the parabola, $x = y^2 - a^2$, and the y axis. Determine the value of a for which the volume obtained by rotating D_a about the y axis is precisely 16 times the volume obtained by rotating D_a about the x axis.

8. A 20 foot telephone pole starts off lying on the ground, and is erected in the following way. First, the base is held stationary. Then one end of a 28 foot cable is attached to the top of the pole, while the other is attached to the back of a tractor which sits 8 feet from the base in the opposite direction. The tractor then begins to drive away from the base at a rate of 3 ft/sec (see diagram below). Assuming that it is approximately high noon, at what rate is the telephone pole's shadow decreasing when the tractor is 12 feet from the base of the pole?



9. The following expression is a rational number. Which one?

$$4\sqrt{3} + \sqrt{129 - 72\sqrt{3}}$$

10. A teacher decides to hold a canned food drive, and asks each of his 29 students to bring in n cans. When the drive is over, the cans completely fill a certain number of boxes which hold exactly 72 cans each, with 3 cans left over. What is the smallest possible value for n?

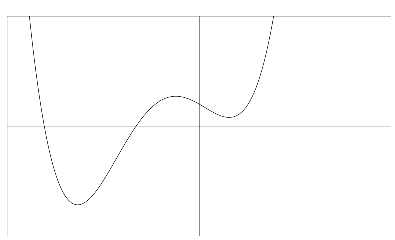
Desserts:

11. A real-valued sequence is defined recursively by $a_0 = 5$ and $a_{n+1} = 6/(8 - a_n)$ for $n \geq 0$. Determine the limit of this sequence, or explain why the limit does not exist.

12. Compute the sum of the infinite series:

$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} = \frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots$$

13. The graph given below is of the function, $y = x^4 + 4x^3 - 2x^2 - 6x + 6$. Find the equation of the unique line which is tangent to this graph in two places.



14. After a lousy April Fool's day with not a single bite, a fisherman is extremely lucky for the remaining 29 days of the month, catching at least one fish each day. When he brags about this fact, and tells the total number of fish caught, a mathematician friend observes that there must have been a continuous stretch of days over which precisely 10 fish were caught. What is the maximum number of fish that the fisherman could have caught for the whole month?

15. If $x \equiv 2$ is one solution to the congruence, $x^{30} \equiv 74 \pmod{125}$ (and it is), how many other solutions are there?

Appetizers:

1. Compute the integral: $\displaystyle\int_0^6 |e^{2x} - 10|\,dx$.

2. A particle moves in the xy-plane so that its position at time t is given by the parametric equations:

$$x(t) = 3\sin\left(\tfrac{2\pi t}{5}\right) - 2\cos\left(\tfrac{2\pi t}{5}\right) \qquad y(t) = \sin\left(\tfrac{2\pi t}{5}\right) + 6\cos\left(\tfrac{2\pi t}{5}\right).$$

When is the particle as far as possible from the origin, and what is the position at those times?

Entrees:

3. Fix a vector $\vec{v}_0 = [x_0\ y_0]^T = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$. Then for $n \geq 0$, define a sequence of vectors $\vec{v}_n$ recursively by

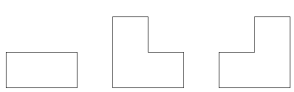
$$\vec{v}_{n+1} = \vec{v}_n + \left(\tfrac{1}{2}\right)^{n+1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{n+1} \vec{v}_0.$$

a. Determine $\vec{v}_4$, if $\vec{v}_0 = [1\ 0]^T$.
b. Find and prove a closed formula for $\vec{v}_n$, if $\vec{v}_0 = [1\ 0]^T$. Then use it to determine $\lim\limits_{n\to\infty} \vec{v}_n$.
c. Find and prove a closed formula for $\vec{v}_n$, if $\vec{v}_0 = [3\ 1]^T$. Then use it to determine $\lim\limits_{n\to\infty} \vec{v}_n$.

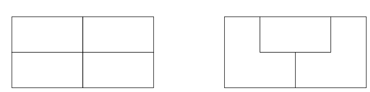
Hint: The recursive formula has a geometric interpretation.

4. a. For a positive integer n, a *composition* of n is a way to write n as the sum of positive integers where order matters. For example, $4 + 3 + 1$ and $3 + 1 + 4$ are two distinct compositions of the number 8. Find a general formula for the number of compositions of n. You must prove your answer for full credit.

b. A $2 \times n$ chessboard is to be covered using dominoes of the three types given below (rotation is **not** allowed). For example, two coverings of the $n = 4$ board are also shown. In general, how many ways are there to cover the $2 \times n$ board using these dominoes? You must prove your answer for full credit.



n=4:



Desserts:

5. The Speedy Racquet Club includes 8 tennis playing members. Two of them, Andy and Bob, are particularly talented. While they are evenly matched against each other, the probability of either winning against some other player from the club is $\frac{2}{3}$. The remaining six members of the club are also evenly matched against each other. The club organizes a tournament with the following rules.

(1) All eight players are randomly paired to play in the first round.
(2) The four winners from the first round are then randomly paired to play in the semi-final round.
(3) The two winners from the semi-final round are then paired to play in the finals.

Compute the probability that Andy beats Bob in the finals. You may assume that the outcomes of separate matches are independent.

6. A certain "spring-mass system" consists of a mass attached to a spring and damping device. When a time-varying external force of $F(t)$ is applied to this system, the displacement of the mass, $x(t)$, can be shown to satisfy the following differential equation.

$$x'' + 6x' + 8x = F(t)$$

a. If the external force is a simple oscillation, given by $F(t) = F_0 \cos(\omega t)$, prove that there will always be a unique solution of the form $x(t) = A\cos(\omega t) + B\sin(\omega t)$. This is called the steady state solution.

b. Suppose, in particular, that the external force is taken to be $F(t) = 80\cos(4t)$, and that $x(t)$ is the steady state solution. What will the maximum displacement of the mass be?