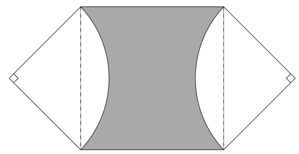
Appetizers:

1. The figure below shows a unit square, partially covered by two quarter-circles. What is the area of the shaded region?



Solution: The radius of each quarter circle is $\frac{1}{\sqrt{2}}$. Hence the area of each sector is given by

$$A_S = \frac{1}{4} \cdot \pi \left(\frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{8} - \frac{1}{4}.$$

So the area of the shaded region is just $1 - 2A_S = \frac{3}{2} - \frac{\pi}{4}$.

2. Write the number 2010 in base 3.

Solution: $2010 = 2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3$. So the answer is $(2202110)_3$.

3. A function $y = f(x)$ is implicitly defined by the equation, $3^y = x^2$, for all positive values of x. Determine the derivative, $\frac{dy}{dx}$, at the point $(9, 4)$.

Solution: If we differentiate implicitly, we find that $3^y \ln 3 \cdot \frac{dy}{dx} = 2x$. Therefore we have

$$\frac{dy}{dx} = \frac{2x}{3^y \ln 3} = \frac{2x}{x^2 \ln 3} = \frac{2}{x \ln 3}.$$

On the other hand, we could solve for y first by taking natural logs: $y = \frac{2 \ln x}{\ln 3}$. Finally, we plug in $(9, 4)$ to get $dy/dx = 2/(9 \ln 3)$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies $f(-5) = 8$ and $f(0) = 2$, and is **even**. Define a new function g by

$$g(x) = \begin{cases} f(x), & \text{if } x \le 0 \\ 4 - f(x), & \text{if } x > 0 \end{cases}.$$

Compute $\int_{-5}^{5} g(x)\,dx$.

Solution: The key is to observe that $h(x) := g(x) - 2$ is now an odd function. Indeed, suppose $x > 0$. Then

$$h(-x) = g(-x) - 2 = f(-x) - 2 = f(x) - 2 = -(2 - f(x)) = -(g(x) - 2) = -h(x)$$

(and $h(0) = 0 = -h(0)$). Since the integral of an odd function from -5 to 5 is 0, we have

$$\int_{-5}^{5} g(x)\,dx = \int_{-5}^{5} (h(x) + 2)\,dx = \int_{-5}^{5} h(x)\,dx + \int_{-5}^{5} 2\,dx = 0 + 20 = 20.$$

5. For distinct primes p and q, and a positive integer n, define the representation number $C_{p,q}(n)$ to be the number of integral solutions (x, y, z) to the equation

$$x^2 + py^2 + qz^2 = n.$$

Compute $C_{3,7}(56)$.

Solutions: There are 5 solutions with $x, y, z \geq 0$, namely

$$\{(1,3,2),(4,2,2),(5,1,2),(1,4,1),(7,0,1)\}.$$

When signs are considered, there are then 36 different possibilities. So $C_{3,7}(56) = 36$.

Entrees:

6. How many solutions does the following equation have over the closed interval $[-2\pi, 2\pi]$?

$$2 - \sin^2\theta = 3\cos\theta - \cos^3\theta$$

Solution: Replacing $\sin^2\theta$ with $1 - \cos^2\theta$, we can rewrite the equation as

$$\cos^3\theta + \cos^2\theta - 3\cos\theta + 1 = 0$$
$$(\cos\theta - 1)(\cos^2\theta + 2\cos\theta - 1) = 0.$$

On one hand, we could have $\cos\theta = 1$, which leads to three solutions: $\theta = -2\pi, 0, 2\pi$. Otherwise, the quadratic formula tells us that $\cos\theta = -1 \pm \sqrt{2}$. The minus is not possible, but the plus leads to unique solutions in the first and fourth quadrants. Thus we have four more solutions in the given range, for a total of 7 solutions.

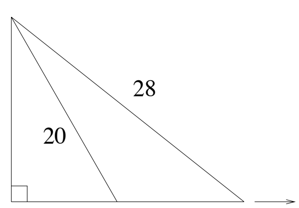
7. Let D_a be the region in the xy-plane which is bounded by the parabola, $x = y^2 - a^2$, and the y axis. Determine the value of a for which the volume obtained by rotating D_a about the y axis is precisely 16 times the volume obtained by rotating D_a about the x axis.

Solution: Let $V_{a,x}$ be the volume of D_a rotated about the x axis, and likewise for $V_{a,y}$. Then

$$V_{a,x} = \int_{-a^2}^{0} \pi(\sqrt{x + a^2})^2\, dx = \pi\Big|_{-a^2}^{0} \tfrac{1}{2}x^2 + a^2x = \tfrac{1}{2}\pi a^4$$
$$V_{a,y} = \int_{-a}^{a} \pi(a^2 - y^2)^2\, dy = \pi\Big|_{-a}^{a} a^4y - \tfrac{2}{3}a^2y^3 + \tfrac{1}{5}y^5 = \tfrac{16}{15}\pi a^5$$

Setting $V_{a,y} = 16V_{a,x}$, we must then have $a = 15/2$.

8. A 20 foot telephone pole starts off lying on the ground, and is erected in the following way. First, the base is held stationary. Then one end of a 28 foot cable is attached to the top of the pole, while the other is attached to the back of a tractor which sits 8 feet from the base in the opposite direction. The tractor then begins to drive away from the base at a rate of 3 ft/sec (see diagram below). Assuming that it is approximately high noon, at what rate is the telephone pole's shadow decreasing when the tractor is 12 feet from the base of the pole?



Solution: Label the distance from the tractor to the base as x, and label the shadow length as y. Then from Pythagorean Theorem (twice), we have

$$(x+y)^2 + (20^2 - y^2) = 28^2$$
$$x^2 + 2xy = 28^2 - 20^2 = 384$$

Now we differentiate implicitly with respect to t, which gives us

$$2x\frac{dx}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0.$$

Finally, we plug in the instantaneous information of $x = 12$, $y = 10$, and $\frac{dx}{dt} = 3$, and find that $\frac{dy}{dt} = -\frac{11}{2}$ ft/sec.

9. The following expression is a rational number. Which one?

$$4\sqrt{3} + \sqrt{129 - 72\sqrt{3}}$$

Solution: Set the expression equal to r, the rational number. Taking the $4\sqrt{3}$ to the other side of the equation, and then squaring both sides, we get

$$129 - 72\sqrt{3} = r^2 + 48 - 8r\sqrt{3}.$$

Now, since the $\sqrt{3}$ is irrational, it follows that $-8r = -72$, and hence $r = 9$.

10. A teacher decides to hold a canned food drive, and asks each of his 29 students to bring in n cans. When the drive is over, the cans completely fill a certain number of boxes which hold exactly 72 cans each, with 3 cans left over. What is the smallest possible value for n?

Solution: The given information tells us that $29n \equiv 3 \pmod{72}$. By the Chinese Remainder Theorem, we may equivalently solve $5n \equiv 3 \pmod 8$ and $2n \equiv 3 \pmod 9$ simultaneously. By constructing multiplicative inverses of 5 and 2, or simply by inspection, this leads to $n \equiv 7 \pmod 8$ and $n \equiv 6 \pmod 9$. Finally, we lift to a solution mod 72 in the standard way to get

$$n \equiv 7(9)(1) + 6(8)(-1) \equiv 15 \mod 72.$$

So the answer is $n = 15$, and we can easily check that $15 \cdot 29 = 435 = 6 \cdot 72 + 3$.

Desserts:

11. A real-valued sequence is defined recursively by $a_0 = 5$ and $a_{n+1} = 6/(8 - a_n)$ for $n \geq 1$. Determine the limit of this sequence, or explain why the limit does not exist.

Solution: The first few terms $(5, 2, 1, \frac{6}{7}, \frac{21}{25}, \dots)$ suggest that (a_n) is monotone decreasing, and bounded below. To prove this, we compute the difference between consecutive terms:

$$a_n - a_{n+1} = \frac{a_n^2 - 8a_n + 6}{a_n - 8} = \left[\frac{a_n - (4+\sqrt{10})}{a_n - 8}\right](a_n - (4 - \sqrt{10})).$$

From this expression, it is clear that if $4 - \sqrt{10} < a_n < 4 + \sqrt{10}$ (which is less than 8), then $4 - \sqrt{10} < a_{n+1} < a_n$. So because our sequence does begin in the desired interval, it follows that (a_n) is monotone decreasing and bounded below by $4 - \sqrt{10}$. Thus, it converges by the Monotone Convergence Theorem. Let $L = \lim a_n = \lim a_{n+1}$. Then, setting the limits of a_{n+1} and $6/(8 - a_n)$ equal, we must have

$$L = \frac{6}{8 - L} \Rightarrow L^2 - 8L + 6 = 0 \Rightarrow L = 4 \pm \sqrt{10}.$$

Since the limit can *not* be $4 + \sqrt{10}$, it must equal $4 - \sqrt{10}$.

12. Compute the sum of the infinite series:

$$\sum_{n=0}^{\infty} \frac{(n+1)^2}{n!} = \frac{1^2}{0!} + \frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \cdots$$

Solution: On the one hand, we could rewrite the numerator, so that cancellation with the $n!$ occurs.

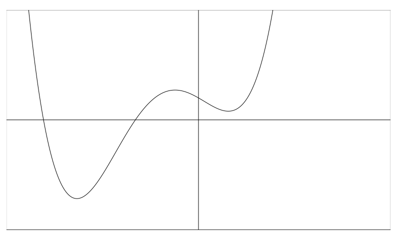
$$\begin{aligned}\sum_{n=0}^{\infty} \frac{[n(n-1)+3n+1]}{n!} &= \sum_{n=0}^{\infty} \frac{n(n-1)}{n!} + 3\sum_{n=0}^{\infty} \frac{n}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} \\ &= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + 3\sum_{n=1}^{\infty} \frac{1}{(n-1)!} + \sum_{n=0}^{\infty} \frac{1}{n!} \\ &= e + 3e + e = 5e\end{aligned}$$

That approach uses only the well-known series representation for the number e. On the other hand, we may begin with the Maclaurin Series for e^x, and then build up to to a generating function for this series as follows.

$$g(x) = [x(xe^x)']' = \frac{1}{0!} + \frac{2^2x}{1!} + \frac{3^2x^2}{2!} + \frac{4^2x^3}{3!} + \cdots$$

We find that $g(x) = (x^2 + 3x + 1)e^x$, and so the series we want is simply $g(1) = 5e$.

13. The graph given below is of the function, $y = f(x) = x^4 + 4x^3 - 2x^2 - 6x + 6$. Find the equation of the unique line which is tangent to this graph in two places.



Solution: Say the line has equation $y = mx + b$. If we let $g(x) = f(x) - (mx + b)$, then $g(x)$ will necessarily be a fourth degree polynomial with repeated roots at two places. Hence it is the square of a quadratic, and

$$x^4 + 4x^3 - 2x^2 - 6x + 6 - (mx + b) = (x^2 + ax + b)^2$$

for some constants a and b. It suffices now to compare x^3 and x^2 terms, to determine that $a = 2$ and $b = -3$. This, in turn, implies that $y = 6x - 3$ is the original line.

14. After a lousy April Fool's day with not a single bite, a fisherman is extremely lucky for the remaining 29 days of the month, catching at least one fish each day. When he brags about this fact, and tells the total number of fish caught, a mathematician friend observes that there must have been a continuous stretch of days over which precisely 10 fish were caught. What is the maximum number of fish that the fisherman could have caught for the whole month?

Solution: If the total number is 49 or more, it is straightforward to construct scenarios in which the fisherman never catches 10 fish over a consecutive span of days. For example, we have

$$49 = (1 + \cdots + 1) + 11 + (1 + \cdots + 1) + 11 + (1 + \cdots + 1),$$

where he catches one fish per day for 9 days in a row (three times). Conversely, if he caught 48 or fewer, we can show the 10 fish span with a pigeonhole argument. Let a_n be the number of fish caught so far on the nth day of April, and consider the numbers:

$$a_1, a_2, a_3, \ldots, a_{30}, (a_1 + 10), (a_2 + 10), (a_3 + 10) \ldots, (a_{30} + 10).$$

Note that (a_n) is a strictly increasing sequence. So here we have 60 numbers which are between 0 and 58 (inclusive). Hence, by the pigeonhole principle, two of the numbers must be the same. But the a_i's are distinct! So this means that $a_j = a_i + 10$ for some $i < j$, and therefore the fisherman has caught exactly 10 fish between days $(i+1)$ and j (inclusive).

15. If $x \equiv 2$ is one solution to the congruence, $x^{30} \equiv 74 \pmod{125}$ (and it is), how many other solutions are there?

Solution: First observe that $125 = 5^3$. So we know that there is a primitive root γ such that $\gamma^{\phi(125)} \equiv 1 \pmod{125}$, where $\phi(125) = 5^3 - 5^2 = 100$. As 2 and 74 are relatively prime to 125, we may write $2 \equiv \gamma^{r_0}$ and $74 \equiv \gamma^s$. Then we have $30r_0 \equiv s \pmod{100}$. From this we learn that s is divisible by $\gcd(30, 100) = 10$, and we let $s = 10s_0$. Now let $x = \gamma^r$ for any r (which represents any solution). Then we have

$$\begin{aligned} x^{30} \equiv 74 \pmod{125} &\Leftrightarrow \gamma^{30r} \equiv \gamma^{10s_0} \pmod{125} \\ &\Leftrightarrow 30r \equiv 10s_0 \pmod{100} \\ &\Leftrightarrow 3r \equiv s \pmod{10} \end{aligned}$$

So the solutions are determined by this class of r mod 10, but that yields 9 more classes mod 100, for a total of 10 solutions to the original congruence. Indeed, here they are:

$$\{2, 23, 27, 48, 52, 73, 77, 98, 102, 123\}.$$

Appetizers:

1. Compute the integral: $\int_0^6 |e^{2x} - 10|\, dx$.

Solutions: The integral must be broken up as

$$\int_0^{\frac{1}{2}\ln 10} (10 - e^{2x})\, dx + \int_{\frac{1}{2}\ln 10}^{6} (e^{2x} - 10)\, dx,$$

which evaluates with a simple u-substitution to $e^{12}/2 + 10\ln 10 - 139/2$.

2. A particle moves in the xy-plane so that its position at time t is given by the parametric equations:

$$x(t) = 3\sin\left(\tfrac{2\pi t}{5}\right) - 2\cos\left(\tfrac{2\pi t}{5}\right) \qquad y(t) = \sin\left(\tfrac{2\pi t}{5}\right) + 6\cos\left(\tfrac{2\pi t}{5}\right).$$

When is the particle as far as possible from the origin, and what is the position at those times?

Solution: One approach is to write the distance from the origin (or better yet, its square) as a function of t. Then set the derivative equal to 0 to find the critical points.

$$\begin{aligned} f(t) = [d(t)]^2 &= (3\sin(\alpha t) - 2\cos(\alpha t))^2 + (\sin(\alpha t) + 6\cos(\alpha t))^2 \\ &= 10\sin^2(\alpha t) + 40\cos^2(\alpha t) \\ f'(t) &= -60\alpha\cos(\alpha t)\sin(\alpha t) = 0 \end{aligned}$$

So critical points are those points where either $\cos(\alpha t) = 0$ (and hence $\sin(\alpha t) = \pm 1$), or vice-versa. But then it's clear from the simplified formula for $f(t)$ which option gives the minimum of 10 and which gives the maximum of 40. So we have a maximum distance of $2\sqrt{10}$ at the points $\pm(2, -6)$, when $\frac{2\pi t}{5} = k\pi$ or $t = \frac{5}{2}k$. Similarly, we have a minimum distance of $\sqrt{10}$ at the points $\pm(3, 1)$, when $t = \frac{5}{4}(2k+1)$.

Entrees:

3. Fix a vector $\vec{v}_0 = [x_0\ y_0]^T = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$. Then for $n \geq 0$, define a sequence of vectors $\vec{v}_n$ recursively by

$$\vec{v}_{n+1} = \vec{v}_n + \left(\tfrac{1}{2}\right)^{n+1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{n+1} \vec{v}_0.$$

a. Determine $\vec{v}_4$, if $\vec{v}_0 = [1\ 0]^T$.
b. Find and prove a closed formula for $\vec{v}_n$, if $\vec{v}_0 = [1\ 0]^T$. Then use it to determine $\lim_{n\to\infty} \vec{v}_n$.
c. Find and prove a closed formula for $\vec{v}_n$, if $\vec{v}_0 = [3\ 1]^T$. Then use it to determine $\lim_{n\to\infty} \vec{v}_n$.

Hint: The recursive formula has a geometric interpretation.

Solution: For Part (a), we plug $\vec{v}_n$ into the formula four times to see that $\vec{v}_0, \ldots, \vec{v}_4$ are equal to

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}, \quad \begin{bmatrix} 3/4 \\ 1/2 \end{bmatrix}, \quad \begin{bmatrix} 3/4 \\ 3/8 \end{bmatrix}, \quad \begin{bmatrix} 13/16 \\ 3/8 \end{bmatrix}.$$

This leads one to conjecture for Part (b) that the components of $\vec{v}_n$ are given by finite geometric series as

$$\begin{aligned} x_{2k} = x_{2k+1} &= 1 - \tfrac{1}{4} + \cdots + \left(-\tfrac{1}{4}\right)^k \\ y_{2k+1} = y_{2k+2} &= \tfrac{1}{2}\left[1 - \tfrac{1}{4} + \cdots + \left(-\tfrac{1}{4}\right)^k\right] \end{aligned}$$

The key to proving this is to realize that the matrix A in the recursive formula satisfies $A^4 = I$. Indeed, it is the $90°$ counter-clockwise rotation matrix. So a straightforward inductive argument can be made in four cases. Once the explicit formula is established, we have

$$\lim_{n\to\infty} \vec{v}_n = \lim_{n\to\infty} \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \left(\frac{1}{1+1/4}\right) \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix}$$

Part (c) is almost identical. In fact, using the notation from above we now have

$$\vec{v}_n = x_n \begin{bmatrix} 3 \\ 1 \end{bmatrix} + y_n \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

So now the limit of $\vec{v}_n$ is obtained by replacing x_n with $\lim x_n = 4/5$ and y_n with $\lim y_n = 2/5$, i.e., $[2\ 2]^T$.

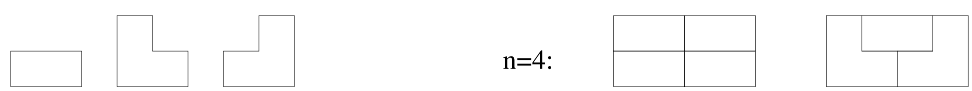
4. a. For a positive integer n, a *composition* of n is a way to write n as the sum of positive integers where order matters. For example, $4 + 3 + 1$ and $3 + 1 + 4$ are two distinct compositions of the number 8. Find a general formula for the number of compositions of n.

Solution: The number of compositions of n is given by $c(n) = 2^{n-1}$. One way to prove this is to use strong induction, and the fact that

$$c(n) = 1 + c(1) + c(2) + \cdots + c(n-1).$$

Another approach is to imagine n objects arranged in a row, with $(n-1)$ spaces in between. Each composition can be viewed as a way to remove some subset of those spaces.

b. A $2 \times n$ chessboard is to be covered using dominoes of the three types given below (rotation is **not** allowed). For example, the two coverings of the $n = 4$ board are also shown. In general, how many ways are there to cover the $2 \times n$ board using these dominoes? You must justify your answer.



Solution: Say that a $2 \times n$ covering is *primitive* if it contains no covering of the $2 \times m$ board with $m < n$. Then there is no covering for n odd, and only one *primitive* covering for each even $n \geq 2$. Since every covering can be broken down into primitives, there is an equivalence between coverings of the $2 \times (2k)$ board and compositions of k (for example the $n = 4$ coverings shown correspond to the compositions $1 + 1$ and 2). Therefore the number of coverings of the $2 \times n$ board, for n even, is given by $2^{(n/2)-1}$.

Desserts:

5. The Speedy Racquet Club includes 8 tennis playing members. Two of them, Andy and Bob, are particularly talented. While they are evenly matched against each other, the probability of either winning against some other player from the club is $\frac{2}{3}$. The remaining six members of the club are also evenly matched against each other. The club organizes a tournament with the following rules.

(1) All eight players are randomly paired to play in the first round.
(2) The four winners from the first round are then randomly paired to play in the semi-final round.
(3) The two winners from the semi-final round are then paired to play in the finals.

Compute the probability that Andy beats Bob in the finals. You may assume that the outcomes of separate matches are independent.

Solution: The probability that Andy and Bob play in Round 1 is 1/7. Assuming this is not the case, and that they both win, the probability that they meet in the second round is 1/3. Assuming that this is not the case, they at least meet in the finals. So then the probability that Andy wins is 1/2. So the final answer is:

$$P = (1 - 1/7)(2/3)^2(1 - 1/3)(2/3)^2(1/2) = 32/567.$$

6. A certain "spring-mass system" consists of a mass attached to a spring and damping device. When a time-varying external force of $F(t)$ is applied to this system, the displacement of the mass, $x(t)$, can be shown to satisfy the following differential equation.

$$x'' + 6x' + 8x = F(t)$$

a. If the external force is a simple oscillation, given by $F(t) = F_0 \cos(\omega t)$, show that there will always be a solution of the form $x(t) = A\cos(\omega t) + B\sin(\omega t)$. This is called the steady state solution.
b. Suppose, in particular, that the external force is taken to be $F(t) = 80\cos(4t)$, and that $x(t)$ is the steady state solution. What will the maximum displacement of the mass be?

Solution: Let $x(t) = A\cos(\omega t) + B\sin(\omega t)$. We want to decide when, if ever, $x(t)$ satisfies the differential equation. So we compute $x'(t)$ and $x''(t)$, and substitute. We find that

$$[(8-\omega^2)A + 6\omega B]\cos(\omega t) + [-6\omega A + (8-\omega^2)B]\sin(\omega t) = F_0\cos(\omega t).$$

Since $\sin(\omega t)$ and $\cos(\omega t)$ are linearly independent functions. This means that the equation holds if and only if A and B satisfy the following system of linear equations.

$$\begin{aligned}(8-\omega^2)A + 6\omega B &= F_0\\ -6\omega A + (8-\omega^2)B &= 0\end{aligned}$$

When we compute the determinant of the coefficient matrix, we see that there is then a unique solution, regardless of the value of ω.
For Part (b), we choose $\omega = 4$ and $F_0 = 80$. This yields the steady state solution, $x(t) = -\cos(4t) + 3\sin(4t)$. Then we can either set the derivative equal to 0 to find the maximal displacement, or rewrite the function as

$$x(t) = \sqrt{10}\left[(-\tfrac{1}{\sqrt{10}})\cos(4t) + (\tfrac{3}{\sqrt{10}})\sin(4t)\right] = \sqrt{10}\cos(4t-\alpha),$$

where α is the unique angle for which $\cos(\alpha) = -1/\sqrt{10}$ and $\sin(\alpha) = 3/\sqrt{10}$. Thus, the maximum displacement equals the amplitude, which is $\sqrt{10}$.